

# BREAKUP OF LIQUID SHEETS AND JETS

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## Notation List

Symbol	Usual meaning	Any exception
$a$	nozzle inner radius	initial disturbance amplitude
$c$	speed of sound	
$d_0$	nozzle diameter	
$d$	displacement from free surface	
$f$	dimensionless free surface displacement	
$\mathbf{g}$	gravitational acceleration	
$h$	dimensionless half sheet thickness	enthalpy
$(\mathbf{i}, \mathbf{j}, \mathbf{k})$	unit vectors in $(x, y, z)$ direction	
$k$	complex wave number in flow direction	
$\mathbf{k}$	complex wave vector	
$l$	radius ratio, thickness ratio	characteristic length
$n$	wave number in $\theta$ -direction	
$\mathbf{n}$	unit normal vector	
$o$	magnitude as small as	
$p$	dimensionless pressure	
$\mathbf{r}$	dimensionless position vector	
$(r, \theta, z)$	dimensionless cylindrical coordinates	

$s$	distance along surface	
$t$	time	
$\mathbf{v} = (u, v, w)$	dimensionless velocity vector in $(x, y, z)$ direction	
$(x, y, z)$	dimensionless Cartesian coordinates	
$A = Re/(WeQ)^{1/2}$	Taylor parameter	
$B_0$	Bonds number = $We/Fr$	
$C_a$	capillary number = $We/Re$	
$C_p$	constant pressure specific heats	
$C_v$	constant volume specific heats	
<b>D</b>	characteristic matrix	
$D$	characteristic determinant	
$Fr = W_0^2/gH_0$	Froude number	
<b>G</b>	inverse Fourier transform of disturbance	
<b>H</b>	position vector of solid surface	amplitude vector
$H_i$	dimensional half $i$ th layer thickness	
$H_0$	characteristic length	
<b>I</b>	identity matrix	Fourier integral
$J = [SH_0/(\rho_1 v^2)]^{1/2}$	Ohnesorge number	
$K$	adiabatic module of elasticity	initial rate of axial stretching
$L$	characteristic length	intact length
$M$	kinematic viscosity ratio	Mach number $W_1/c_2$
$Ma = S, T T_0/\rho_1 U_0^2 H_0$	Marangoni number	
$M_i$	Mach number in fluid $i$	
$N = \bar{N}_2 = \mu_r$	dynamic viscosity ratio	
$N_j = \nu_j/\nu_1$	kinematic viscosity ratio	

$\bar{N}_\alpha = \mu_\alpha / \mu_1$	dynamic viscosity ratio	
$O$	magnitude as large as	
$Oh = [SH_0 / (\rho_1 v_1^2)]^{1/2} = J$	Ohnesorge number	
$P$	pressure	
$Q = Q_2 = \rho_r$	gas to liquid density ratio	
$\dot{Q}$	volumetric flow rate	
$(R, \theta, Z)$	dimensional cylindrical coordinate	
$Re = \rho_1 U_0 H_0 / \mu_1$	Reynolds number	
$St_0 = Re / Fr$	Stokes number	
$S$	surface tension	
$T$	temperature	period of oscillation
$U_0$	characteristic velocity	
$\mathbf{V} = (U, V, W)$	dimensional velocity vector	
$We = \rho_1 U_0^2 H_0 / S$	Weber number	
$\alpha$	wave number in flow direction	
$\beta$	wave number in direction perpendicular to flow	
$\gamma = C_p / C_v$	specific heat ratio	swirl number = $\Gamma / R_0 W_1$
$\delta$	Dirac delta function	
$\varepsilon$	small parameter	
$\tau$	dimensionless time	
$\tau$	deviatoric stress	
$\zeta$	dimensionless sheet or jet thickness	
$\eta$	dimensionless free surface displacement	
$\theta$	azimuthal angle	phase angle, spray angle
$\kappa$	mean curvature	
$\lambda$	wavelength	
$\mu$	dynamic viscosity	
$\nu$	kinematic viscosity	

$\rho$	density	
$\psi$	stream function	
$\phi$	velocity potential	amplitude of $\psi$
$\sigma$	stress tensor	
$\omega = \omega_r + i\omega_i$	complex wave frequency	dimensional frequency
$\Omega$	dimensional frequency	
$\nabla$	gradient operator	
$\Gamma$	circulation	
Superscripts		
$\bullet$	time rate of change	
T	transpose	
'	perturbation	differentiation
$\wedge$	amplitude	
Subscripts		
adj	adjoint	
,	partial differentiation	
$i$	inner surface	$i$ th interface
$o$	reference quantity	outer surface
$A, B$	fluids $A, B$	
1, 2, 3	fluids 1, 2, 3	
$\alpha$	$\alpha$ th layer	
$l$	liquid	
$g$	gas	

# 1

## Introduction

### 1.1. Overview

When a dense fluid is ejected into a less dense fluid from a narrow slit whose thickness is much smaller than its width, a sheet of fluid can form. When the fluid is ejected not from a slit but from a hole, a jet forms. The linear scale of a sheet or jet can range from light years in astrophysical phenomena (Hughes, 1991) to nanometers in biological applications (Benita, 1996). The fluids involved range from a complex charged plasma under strong electromagnetic and gravitational forces to a small group of simple molecules moving freely with little external force. The fluid sheet and jet are inherently unstable and breakup easily. The dynamics of liquid sheets was first investigated systematically by Savart (1833). Platou (1873) sought the nature of surface tension through his inquiry of jet instability. Rayleigh (1879) illuminated his jet stability analysis results with acoustic excitation of the jet. In some modern applications of the instability of sheets and jets, it is advantageous to hasten the breakup, but in other applications suppression of the breakup is essential. Hence knowledge of the physical mechanism of breakup, aside from its intrinsic scientific value, is very useful when one needs to exploit the phenomenon to the fullest extent. Recent applications include film coating, nuclear safety curtain formation, spray combustion, agricultural sprays, ink jet printing, fiber and sheet drawing, powdered milk processing, powder metallurgy, toxic material removal, and encapsulation of biomedical materials. Current applications can be found in the annual or biannual conference proceedings of several professional organizations, such as the International Conference on Liquid Atomization and Spray Systems (ICLASS) and the Institute for Liquid Atomization and Spray Systems (ILASS) organizations in the Americas, Europe, and Asia, and European and American Coatings Conferences.

Because of the diverse applications, books on the subject tend to focus on specific applications. For example, the book by Lefebvre (1989) centers

around internal combustion, and that of Masters (1985) focuses on powdered milk formation. Intended for immediate practical applications, these books rely heavily on phenomenological correlations. The book by Yarin (1993) provides a mathematical treatment of recent applications involving non-Newtonian fluids. In contrast, this book deals exclusively with Newtonian fluids, which are encountered in most of the known applications. It does not cover such topics as atomization and emulsification of liquid in liquid (Kitamura and Takahashi, 1986; Grandzol and Tallmadge, 1973; Villermaux, 1998; Richards, Beris, and Lenhoff, 1993). Electromagnetic effects on the jet breakups (Balachandran and Bailey, 1981), or the electromagnetic effects on atomization and drop formation (Bailey, 1998; Fenn et al., 1989).

We address first the issue of the origin of the breakup or the physical reasons for the breakup. Therefore the mathematical tool used is linear stability analysis, which predicts the onset of jet and sheet instability. The disturbance consisting of all Fourier components is allowed to grow both spatially and temporally in the sheet or jet flows. If only the classical temporally growing disturbance is considered, one arrives at a paradoxical situation as illustrated in the first section of the next chapter. The onset of instability appears to largely dictate the ultimate outcome of the breakup, as exemplified by Rayleigh's linear stability analysis of a liquid jet. However, the detailed process leading to the eventual breakup requires nonlinear theories to describe. Nonlinear descriptions are given in Chapter 11. The results related to the last stage of breakup and topics that still need further development will be addressed in the Epilogue.

## 1.2. Governing Equations

The governing equations and the corresponding boundary conditions listed below will be referred to in subsequent chapters. Their derivation can be found in standard text books, some of which are given at the end of the chapter. The same notation will be used to denote the same physical variable throughout the book, with few exceptions. When such exceptions on notation take place they will be pointed out; otherwise the same symbol will not be redefined after its first appearance. A list of notations is provided at the front of the book.

Newton's second law of motion applied to a fluid particle gives

$$\rho \frac{D\mathbf{V}}{Dt} = \mathbf{g} + \nabla \cdot \boldsymbol{\sigma}, \quad (1.1)$$

$$\frac{D\mathbf{V}}{Dt} \equiv \mathbf{V}_{,t} + \mathbf{V} \cdot \nabla \mathbf{V},$$

where  $\rho$  is the fluid density,  $\mathbf{V}$  is the velocity vector, and  $t$  is the time. The subscript variable following a comma signifies partial differentiation with that variable,  $D/Dt$  is the substantial derivative as defined,  $\nabla$  is the gradient operator,  $\mathbf{g}$  is the gravitational acceleration, and  $\boldsymbol{\sigma}$  is the stress tensor. For an incompressible Newtonian fluid

$$\boldsymbol{\sigma} = -P\mathbf{I} + \mu[\nabla\mathbf{V} + (\nabla\mathbf{V})^T], \quad (1.2)$$

where  $\mathbf{I}$  is the identity matrix,  $\mu$  is the dynamic viscosity,  $P$  is the pressure, and the superscript  $T$  denotes transpose.

The conservation of mass requires

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{V} = 0. \quad (1.3)$$

For an incompressible fluid  $D\rho/Dt = 0$ , and (1.3) is reduced to

$$\nabla \cdot \mathbf{V} = 0. \quad (1.4)$$

Equations (1.1) to (1.4) are valid for each fluid involved in a flow. The  $i$ -th interface between two adjacent fluids is infinitesimally thin and is mathematically defined by a function  $F_i(\mathbf{r}, t) = 0$ ,  $\mathbf{r}$  being the position vector. The balance of forces exerted on a unit area of interface gives

$$S_i\nabla \cdot \mathbf{n} + [\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}]_{B_i}^{A_i} + \nabla'' S_i = 0, \quad (1.5)$$

where  $S$  is the interfacial tension,  $\mathbf{n}$  is the surface unit normal vector positive if pointed from fluid  $B_i$  to fluid  $A_i$  on the opposite side,  $\nabla''$  is the surface gradient operator, and

$$\begin{aligned} [\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}]_{B_i}^{A_i} &\equiv \mathbf{n}_i \cdot \boldsymbol{\sigma}_{A_i} \cdot \mathbf{n}_i - \mathbf{n}_i \cdot \boldsymbol{\sigma}_{B_i} \cdot \mathbf{n}_i, \\ \mathbf{n}_i &= \nabla F_i / |\nabla F_i|. \end{aligned}$$

For viscous fluids, the kinematic condition at the interface is

$$[\mathbf{V}]_{B_i}^{A_i} = 0, \quad (1.6)$$

$$W_i = \frac{DF_i}{Dt}, \quad (1.7)$$

where  $W_i$  is the component of the  $i$ -th interfacial velocity in the direction in which the distance  $F_i$  from a reference position to the interface is measured. If a fluid is inviscid, then (1.6) does not hold, and (1.7) must be applied for each fluid separately. A viscous fluid sticks to a nonpermeable solid surface, and thus  $\mathbf{V} = 0$  at the solid-viscous fluid interface. If the fluid is inviscid, then it is allowed to slide along the solid surface, but is not allowed to penetrate it. Derivations of Equations (1.1) to (1.7) can be found in the books on

fundamental fluid mechanics cited in the references section at the end of the chapter. Note that non-Newtonian fluids as well as more general interfacial conditions allowing phase changes to take place are not treated in this work.

### 1.3. Dimensionless Parameters

Even for simple Newtonian fluids, the number of dimensionless groups involved in interfacial fluid dynamics is relatively large. To bring out the relevant dimensionless parameters, we nondimensionalize the governing differential system. Identifying the characteristic velocity and length with  $U_0$ , length with  $H_0$ , time with  $H_0/U_0$ , and stress with  $\rho_1 U_0^2$ , where  $\rho_1$  is the density of the fluid designated by subscript 1, we have the following dimensionless governing equations for incompressible Newtonian fluids:

$$Q_i \frac{D\mathbf{v}_i}{D\tau} = \frac{Q_i}{Fr} - \nabla p_i + \frac{N_i}{Re} \nabla^2 \mathbf{v}_i, \quad (1.8)$$

$$\nabla \cdot \mathbf{v}_i = 0, \quad (1.9)$$

kinematic interfacial condition,

$$w_i = h_{i,\tau} + \mathbf{v}_i \cdot \nabla h_i, \quad h_i = F_i/H_0,$$

dynamic interfacial condition,

$$We_i^{-1} \nabla \cdot \mathbf{n}_i = [\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}]_{\beta_i}^{A_i}, \quad (1.10)$$

and the no-slip condition at the solid wall at  $\mathbf{H}/H_0$ , where  $\mathbf{H}$  is the position vector defining the solid wall. The lower case letters are used to denote dimensionless variables corresponding to their dimensional counterparts expressed in capital letters, except for  $\tau$  and  $\boldsymbol{\tau}$ , which are dimensionless time and stress respectively. The dimensionless groups revealed in these equations are

$$\begin{aligned} \text{density ratio } Q_i &= \rho_i/\rho_1, \\ \text{viscosity ratio } N_i &= \mu_i/\mu_1, \\ \text{Reynolds number } Re &= \rho_1 U_0 H_0/\mu_1, \\ \text{Froude number } Fr &= \rho_1 U_0^2/g H_0, \\ \text{Weber number } We &= \rho_1 U_0^2 H_0/S, \\ \text{geometric parameters } &\mathbf{H}/H_0, H_i/H_0. \end{aligned} \quad (1.12)$$

The interface is considered to be homogeneous, otherwise Marangoni numbers associated with  $\nabla \cdot S$  in (1.5) will arise. The interface is also assumed to be isotropic. The quantitative sensitivity of the dynamics of the flow to the variation of these dimensionless groups will be used to reveal the relative

importance of shear, inertial, body, and surface forces in various modes of interfacial instabilities.

### Exercises

- 1.1. Show that if temperature varies along an interface, the surface gradient term in (1.5) leads to the temperature Marangoni number  $Ma = S_{,T} T_0 / \rho U_0^2 H_0$ , where  $S_{,T}$  is the change of surface tension per unit change of temperature,  $T_0$  is a reference temperature, and  $U_0$  is a characteristic velocity. If the fluids on both sides of the interface are stationary, what is the relevant expression for  $U_0$ ?
- 1.2. If the solute concentration varies along an interface, find the expression of the solute Marangoni number.
- 1.3. Show that the Bond number  $B_0 = We / Fr$ , the capillary number  $C_a = We / Re$ , and the Stokes number  $St_0 = Re / Fr$  represent respectively the ratios of body force to surface force, viscous force to surface force, and body force to viscous force.
- 1.4. Show that if  $U_0 = 0$ , the Ohnesorge number  $\equiv [SH_0 / (\rho v^2)]^{1/2}$  is a parameter representing the ratio of the surface force to the viscous force.
- 1.5. Show that the mean curvature  $\nabla \cdot \mathbf{n}$  in (1.10) at a point on a surface  $z = h(x, y, \tau)$  in the Cartesian coordinate  $(x, y, z)$  is given by

$$\nabla \cdot \mathbf{n} = -\frac{h_{,xx} + h_{,yy}}{(1 + h_{,x}^2 + h_{,y}^2)^{3/2}}. \quad (1.11)$$

- 1.6. Show that the mean curvature of a surface  $r = h(z, \theta, t)$  is given by

$$\begin{aligned} \nabla \cdot \mathbf{n} = & \frac{1}{q^2} (h_{,z} q_{,z} + h_{,\theta} q_{,\theta} / h^2) \\ & + \frac{1}{q} \left( \frac{1}{h} + h_{,\theta}^2 / h^3 - h_{,zz} - h_{,\theta\theta} / h^2 \right), \end{aligned} \quad (1.12)$$

where  $q = [1 + (h_{,\theta} / h)^2 + h_{,z}^2]^{1/2}$ .

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